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Title: Towards a nonclassical metatheory for substructural approaches to paradox.

Abstract: It is fairly common to make a distinction between two views about how our semantic theories should be like. One view -the so-called *orthodox* view- is that theories that purport to explain how semantic concepts of a certain language work should be couched in a richer language containing expressions for those semantic concepts. Usually, this is cashed-out in terms of the distinction between an object language and a metalanguage. Alfred Tarski famously pointed out that in order to explain how truth behaves in a certain language we need to (and should) ascend to an essentially richer metalanguage.

A relatively more recent view -usually called the *naive* view- endorses the idea that semantic theories should be couched in a language with enough resources to express its own semantic concepts. In this view there is no need to look for a richer metalanguage or to make an artificial distinction between different languages. The formal languages of our semantic theories should mimic ordinary languages, at least to the extent that in them we can talk about the semantic concepts that apply to the expressions of that same language.

The main idea behind the naive approach is to develop theories about naive semantic concepts. However, although there is a wide consensus on what counts as a naive concept of truth, I think that the situation for the concept of validity is somewhat different. Usually, a validity predicate of a theory $\mathcal{S}$ is said to be naive if the following condition holds:

1. For every $\mathcal{S}$-valid argument from Γ to ϕ, $\mathcal{S}$ should prove the statement expressing that the argument from Γ to ϕ is valid.

Now, although this condition is certainly necessary for the corresponding validity predicate to be naive, it is far from being sufficient. In particular, a naive validity predicate for a theory $\mathcal{S}$ should capture at least one additional feature of $\mathcal{S}$:

2. For every $\mathcal{S}$-*invalid* argument from Γ to ϕ, $\mathcal{S}$ should *disprove* the statement expressing that the argument from Γ to ϕ is valid.

The main goal of this paper is to evaluate whether certain substructural theories are able to characterize what might be viewed as a naive concept of validity in the sense above without falling prey to the any self-referential paradoxes.

There is a small but very interesting literature on how to provide proof procedures for the set of invalidities of classical propositional logic. These proof procedures are designed to prove a certain argument if and only if the argument is *not* valid. As with sets of validities, this task can be done in different ways. Both Caicedo (1978) and Varzi (1990, 1992) offer an axiomatic calculus, while Tiomkin (1988) and Carnielli & Pulcini (2016) provide a sequent calculus.

Now, the sort of sequent calculus we need does not only work with sequents properly speaking, for the proof of a sequent is meant to represent that the corresponding argument is valid. What we need are, in addition, antisequents. Following Tiomkin (1988) we say that an *antisequent* is an object of the form $\Gamma \not\Rightarrow \Delta$. Intuitively, we should understand this as 'the inference from Γ to Δ fails'.

What does a sequent calculus for invalidities look like? Since we want to talk about both validities and invalidities, what we need is a mixed system. In such a system, there will be two types of objects, sequents and antisequents, and some of the rules will allow us to go from one type of object to the other. Our full system (which I'll call $\mathbb{M}$ for 'mixed') is then as follows:

Definition (*The system* $\mathbb{M}$) Let Γ, Δ, Π and Σ be (finite) multisets of formulas, and let ϕ and ψ be formulas. The system $\mathbb{M}$ is given by the following initial sequents, initial antisequents and rules:

$$\textit{Axioms}\ \frac{}{p \Rightarrow p} \qquad \textit{Antiaxioms}\ \frac{}{p_1, ..., p_n \not\Rightarrow q_1, ..., q_m}\ \text{where for all } i, j\ p_i \neq q_j$$

$$\textit{Weak}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta'}\ \text{where } \Gamma \subseteq \Gamma' \text{ and } \Delta \subseteq \Delta'^{1} \qquad \textit{Antiweak}\ \frac{\Gamma' \not\Rightarrow \Delta'}{\Gamma \not\Rightarrow \Delta}\ \text{where } \Gamma \subseteq \Gamma' \text{ and } \Delta \subseteq \Delta'$$

$$\text{L}\textit{Contr}\ \frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} \qquad \text{L}\textit{AntiContr}\ \frac{\Gamma, \phi \not\Rightarrow \Delta}{\Gamma, \phi, \phi \not\Rightarrow \Delta}$$

$$\text{R}\textit{Contr}\ \frac{\Gamma \Rightarrow \phi, \phi, \Delta}{\Gamma \Rightarrow \phi, \Delta} \qquad \text{R}\textit{AntiContr}\ \frac{\Gamma \not\Rightarrow \phi, \Delta}{\Gamma \not\Rightarrow \phi, \phi, \Delta}$$

$$\text{L}\neg^{+}\ \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma, \neg\phi \Rightarrow \Delta} \qquad \text{L}\neg^{-}\ \frac{\Gamma \not\Rightarrow \phi, \Delta}{\Gamma, \neg\phi \not\Rightarrow \Delta}$$

$$\text{R}\neg^{+}\ \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\phi, \Delta} \qquad \text{R}\neg^{-}\ \frac{\Gamma, \phi \not\Rightarrow \Delta}{\Gamma \not\Rightarrow \neg\phi, \Delta}$$

$$\text{L}\wedge^{-}\ \frac{\Gamma, \phi, \psi \not\Rightarrow \Delta}{\Gamma, \phi \wedge \psi \not\Rightarrow \Delta}$$

$$\text{L}\wedge^{+}\ \frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta} \qquad \text{R}\wedge^{-}\ \frac{\Gamma \not\Rightarrow \phi, \Delta}{\Gamma \not\Rightarrow \phi \wedge \psi, \Delta}$$

$$\text{R}\wedge^{+}\ \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta} \qquad \text{R}\wedge^{-}\ \frac{\Gamma \not\Rightarrow \psi, \Delta}{\Gamma \not\Rightarrow \phi \wedge \psi, \Delta}$$

The system so far is nothing more than a mixed version of classical propositional logic. If we take the positive part only, we obtain the set of validities of classical propositional logic, and if we take the negative part only, we obtain the set of invalidities of classical propositional logic.

Now, if we want to talk about (and prove things having to do with) the validity and invalidity of inferences, it is not enough to consider purely positive

[1] Inclusion between multisets should be understood as usual. Also, in the rules *Weak* and *Antiweak* I assume that at least one of the two inclusions has to be strict.

or purely negative rules. As I mentioned before, we want to say, for instance, that if q does not follow from p, then $\neg Val(p,q)$. This motivates the presence of mixed rules. The validity rules are then as follows:

$$\text{R}Val^{+}\ \frac{Val(\Gamma_1,\Delta_1),...,Val(\Gamma_k,\Delta_k),\phi_1,\ldots,\phi_n \Rightarrow \psi_1,\ldots,\psi_m}{Val(\Gamma_1,\Delta_1),...,Val(\Gamma_k,\Delta_k) \Rightarrow Val(\Phi,\Psi)}$$

$$\text{L}Val^{+}\ \frac{\phi_1,...,\phi_n \not\Rightarrow \psi_1,...,\psi_m}{Val(\Phi,\Psi) \Rightarrow}$$

$$\text{L}Val^{-}\ \frac{\phi_1,...,\phi_n \Rightarrow \psi_1,...,\psi_m}{Val(\Phi,\Psi) \not\Rightarrow}$$

$$\text{R}Val^{-}\ \frac{\Gamma,\phi_1,...,\phi_n \not\Rightarrow \psi_1,...,\psi_m,\Delta}{\Gamma \not\Rightarrow Val(\Phi,\Psi),\Delta}$$

I'll call the resulting system $\mathbb{M}^V$. The crucial feature of $\mathbb{M}^V$ is that we can now talk about invalidity quite straightforwardly. Moreover, the resulting theory is complete in two ways: for every $\mathcal{S}$-valid inference from Γ to Δ, we have $\Rightarrow Val(\Gamma,\Delta)$ and for every $\mathcal{S}$-invalid inference from Γ to Δ we have $Val(\Gamma,\Delta) \Rightarrow$.

One of the things we would like to know about our system is if it is consistent. Since our proof system has two sorts of objects -sequents and antisequents- we can define a new property which I'll call *external consistency*. I'll say that a system is *externally consistent* if it is not the case that $\Gamma \Rightarrow \Delta$ and $\Gamma \not\Rightarrow \Delta$ are both provable in the system.

Theorem 0.1 (External consistency) *$\mathbb{M}^V$ is externally consistent. That is, for any Γ and any Δ it is not the case that $\Gamma \Rightarrow \Delta$ and $\Gamma \not\Rightarrow \Delta$ are both provable in $\mathbb{M}^V$.*

There are still several open questions remaining. Of particular interest is the issue of the self-referential paradoxes and more specifically the so-called v-Curry paradox. This creature reemerges in this setting with a different face. One remarkable feature of this paradox is that it seems to affect non-contractive theories. In this sense, the paradox is quite similar to the paradoxes of logical properties introduced by Zardini in (2013) and (2014).

In spite of this, the non-contractivist theorists has, in my opinion, a plausible way out. She can embrace a conception of validity where certain inferences are neither valid nor invalid, i.e. a conception of validity where there could be validity gaps. Whether this is an unaffordable cost is a question that is too difficult to discuss here.

References

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