An Embedding of First-Order Modal Logic with Admissible Semantics into Many Sorted Logic

1 Introduction

The expressive power of modal propositional languages has been investigated in detail by 'modal correspondence theory' and, as a result, its relationship to the expressivity of first–order and second–order languages may pass as well–understood. In contrast, the situation for modal predicate logic (QML) is completely different. Extending modal correspondence theory to quantified modal logic proved to be much more difficult [6]. In consequence of this impediment, the relation between first–order modal languages and their extensional counterparts - this time two-sorted first-order languages with variables for both individuals and worlds - has been less explored. Moreover, it is a fact that first–order modal languages occur almost everywhere. Especially in several philosophical disciplines they are used as an important technical instrument.

This work will be concerned with the translation of quantified modal logic embedded with a general semantic (*QML-G*) -a kind of admissible semantics- [5], [4], into a many-sorted language. Our proposal follows the method of Maria Manzano, [3], [2], where the signature of the logic *XL* under consideration is transformed into a many-sorted structure, by translating the expressions of *XL* to the many-sorted logic *MSL* and the very structures of *XL* are changed to many-sorted structures.

2 Quantifiers in QML-G

Suppose that a formula A has at least one variable free x. In QML-G the formula $\forall xA$ is true in the world a if and only if there exist some proposition X such that X entails every instantiation of A and X holds in a.

A proposition is a set of worlds and it holds in a world, if this world is in the proposition. A proposition X entails a formula A , if A is true in every world in X.

A Quantified General Frame (frame-QG) $\mathcal{F} = \langle W, R, I, \mathit{Prop}, \mathit{FunProp}\rangle$ is an structure in which W is a non empty set, R is a binary relation on W, I is a nonn empty set (of "individuals"), *Prop* is a set of subsets of W, and *FunProp* is a set of ***proposicional functions***, i.e., functions from I^ω to *Prop*, such that they satisfy certain closure conditions

$\mathcal{M}$, $a \vDash A$ is a truth / satisfaccin relation between worlds $a \in W$, assignation to the variables $f \in I^\omega$ and formulas A. To each frmula a *truth set* $\left|A\right|_f^{\mathcal{M}} =_{def} \{b \in W : \mathcal{M}, b \vDash_f A\}$ is associated. Concerning the quantifier we have:

- $\mathcal{M}$, $a \vDash_f \forall x_n A$ iff there is an $X \in \mathit{Prop}$ such that $X \subseteq \bigcap_{j\in n} \left|A\right|_{f\left[j/I\right]}^{\mathcal{M}}$ y $a \in X$.

These properties, and the closure properties of *FunProp*, guarantee that always $\left|A\right|^{\mathcal{M}} \in$ *FunProp* and hence, $\left|A\right|_f^{\mathcal{M}} \in \mathit{Prop}$, for all formula A. The satisfaction clause for $\forall x_n$ can be stated as follows:

$$\left|\forall x_n A\right|_f^{\mathcal{M}} = \prod_{j\in I} \left|A\right|_{f\left[j/n\right]}^{\mathcal{M}}$$

showing that $\left|\forall x_n A\right|_f^{\mathcal{M}}$ is the least upper bound of $\left\{\left|A\right|_{f\left[j/n\right]}^{\mathcal{M}} : j \in I\right\}$ in the partial order set $\left(Prop, \subseteq\right)$. This is the natural interpretation of $\forall$ in algebraic semantics. To obtain the standard Tarskian semantics for $\forall$ we need that this least upper bound was $\bigcap_{j\in I}\left|A\right|_{f\left[j/n\right]}^{\mathcal{M}}$. But this is not necessarily so.

3 Modal Logic in Many-Sorted Logic

The expressive power of predicate modal language can be studied by using the standard techniques for modal propositional logics. In this case, a two-sorted language (with variables for individuals and worlds), is used to establish the correspondence between logics; a predicate interpreting the accessibility relation and predicates of *n+1* arity, which correspond to each predicate of arity n of the predicate modal language.

3.1 Standard translation

The standard translation expresses in first order the truth conditions for modal formulas .

4 The formal many-sorted language MSL

Now let us briefly describe, in accordance with [2], the formal language we will consider, bearing in mind that we want to translate **QML-G** into this logic.

4.1 The signature $\Sigma^{\blacklozenge}$

Let $SORT = \{0, 1, 2\}$.

The signature $\Sigma^{\blacklozenge} = \langle SORT, FUNC^{\blacklozenge}\rangle$ is such that:

1. **SORT** is the set defined above.

2. $FUNC^{\blacklozenge}$ is a function which takes as arguments the elements of $OPER.SYM^{\blacklozenge}$ consisting of: **REL.CONS** of the quantified modal logic, the equality among individuals of sorts 1 (worlds) and 2 (individuals), i.e., E_1 and E_2, and the connectives. All the values of this function are as usual but

   $FUNC^{\blacklozenge}(P) = \langle 0, 2, \overset{n}{\ldots}, 2, 1\rangle$

5 Adding propositions

Our first purported modification for this language concerns the set of types, following ideas of Gallin [1], but without explicitly appealing to an intensional type. Thus, we can go by adding propositions, which are the kind of things expressed by a formula A, when its variables are instantiated.

As a result, we can define the set $PROP \subseteq 2^{\mathrm{D}_1}$ as a non-empty set of propositions in the standard sense. Consequently, if PROP is always a non-empty set of propositions, we count now with a devise to define in the many-sorted structure what is a proposition and what is a set of admissible propositions. Propositional function can be introduced also in this way.

References

[1] Gallin D. *Intensional and higher-order modal logic.* North-Holland, Amsterdam, 1975.

[2] Manzano M. *Extensions of First-Order Logic.* Cambridge University Press, London, 2005.

[3] Manzano M. Divergencia y rivalidad entre lógicas (in spanish). In *Orayen R. and Moretti A. (eds.), Filosofía de la Lógica. Enciclopedia Iberoamericana de Filosofía*, volume 27. Spain: Trotta, 2006.

[4] Goldblatt R. *Quantifiers, Propositions and Identity: Admissible Semantics for Quantified Modal and Substructural Logics.* Cambridge University Press, London, 2002.

[5] Goldblatt R and Mares E. D. Frame correspondence in modal predicate logic. In *Governatori G., Hodkinson I. and Venema Y.(eds.): Advances in Modal Logic*, volume 6. London: College Publications, 2006.

[6] van Benthem J. Frame correspondence in modal predicate logic. In *Feferman S. and Sieg W. (eds.): Proofs Categories and Computations. Essays in Honor of Grigori Mints.* London: College Publications, 2010.