A PARACONSISTENT ROUTE TO SEMANTIC CLOSURE

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Abstract. In this paper we present a non-trivial and expressively complete paraconsistent naïve theory of truth, as a step in the route towards semantic closure. We achieve this goal by expressing self-reference with a *weak* procedure, that uses equivalences between expressions of the language, as opposed to a strong procedure, that uses identities. Finally, we make some remarks regarding the sense in which the theory of truth discussed has a property closely related to functional completeness, and we present a sound and complete three-sided sequent calculus for this expressively rich theory.

1. Introduction

A *semantically closed* theory is a theory that is capable of expressing every semantic concept –and, hence, every semantic concept applicable to the theory itself. If this is the case, then it is reasonable to think that every *extensional* (i.e. definable by a truth-table) notion should be expressible in the theory. This phenomenon is indeed closely related with the well-known notion of functional completeness. Thus, it seems that a *necessary* requirement of a semantically closed theory is for it to have something to do with this sort of completeness.

Moreover, if a theory is semantically closed, as a consequence, it will be able to express its own concept of *truth*. In what follows, we will refer to theories that are able to achieve this goal as *naïve* theories of truth.[1] *The aim of this paper is to discuss whether or not its possible to get functionally complete naïve theories of truth, as a necessary step in the route towards semantic closure.* Were this project to fail, then the general endeavor of having a semantically closed language will be doomed.

Now, in the context of a naïve theory of truth, we will say that the corresponding truth predicate behaves like a naïve truth predicate. But, then, what is required for a truth predicate to deserve being called naïve? The first reasonable desideratum we will be considering, is for it to have a *transparent* truth predicate. Let A be a sentence of the language, let Tr be a truth predicate and let $\langle A \rangle$ be a name for the formula A. Then Transparency requires that the sentence A and $Tr(\langle A \rangle)$ should be intersubstitutable *salva veritate* in every non-opaque context.[2] In extensional semantics, transparency is granted if for every valuation v, $v(A) = v(Tr(\langle A \rangle))$. Transparency is usually understood as one of the most central features of a naïve theory of truth. In fact, the consensus around this point is vast.[3]

Key words and phrases. Truth, Semantic Closure, Paradoxes, Paraconsistency, Self-Reference.

[1]As opposed to the orthodox view on formal theories of truth (incarnated e.g. by Tarski [28]), one of the main advantages of a naïve theories of truth is that they avoid hierarchical approaches to the concept truth.

[2]I.e. in every context where both A, or $Tr(\langle A \rangle)$, are *used*, and not *mentioned*.

[3]With the notable exception of Priest, who rejects it in [22].

The second desideratum we will be considering, demands the truth predicate to validate the naïve (or *full*) T-Scheme. The T-Scheme expresses the idea that, for any sentence A, the assertion of A is equivalent to the assertion that A is true.[4]

$$A \Leftrightarrow Tr(\langle A \rangle)$$

It should be noticed that not all non-hierarchical theories of truth validate the T-Scheme, most authors recognize this as a flaw of their approaches.[5] Notice that in the above expression $\Leftrightarrow$ is *some* biconditional. Considering these two first desiderata, a natural question emerges: how should the biconditional featured in the T-Scheme behave, in order to fulfill both of these requirements? As is well known, not all the options are well-suited. For example, Tarski showed in [28] that he two-valued biconditional of classical logic will not do the trick.

A general remark regarding the alternatives for this biconditional is provided in [13] by Laura Goodship. If we want the theory to be safe from trivialization due to semantic paradoxes, there seems to be two main routes: (1) either the conditional should invalidate *Modus Ponens*, or (2) the conditional should invalidate *Contraction* and *Pseudo Modus Ponens*. In this paper, we will explore one possible realization of the first alternative[6]: we will consider naïve theories of truth that invalidate *Modus Ponens* due to the fact that they are based on paraconsistent[7] –and, therefore, non-classical– logics.[8]

Even if naïve truth predicates are usually taken to force the move to non-classical (e.g. paraconsistent) logics, most people working in non-classical theories of truth agree that classical logic is a fine logic to deal with e.g. consistent contexts or sentences. Thus, it seems to be a desirable third desideratum to have the linguistic and logical means to *recover* or *recapture* classical logic, for those areas where e.g. consistency is granted. We will do this with the help of techniques developed by logicians working in logics of formal inconsistency (LFIs).[9], which will help us to have linguistic resources to distinguish classical from non-classical contexts.[10]

Finally, if having naïve theories of truth is a necessary step in the route towards semantic closure, then it also seems reasonable to require naïve theories of truth

[4]The T-Scheme was proposed by Tarski as a very basic feature of the truth predicate. In this paper, we will be agreeing with the received view in recognizing the T-Scheme as a central feature of a properly speaking *naïve* truth predicate.

[5]See, for example, Kripke [16], Halbach [14]. Notice that Field [11] devotes an entire work to find a way to extend Kripke's theory of truth with a *new* (bi)conditional that validates the T-Scheme.

[6]Additionally, there are in fact many connections between our project and the one presented in [13] by Laura Goodship. Those connections will become explicit when we give a few hints about how we get self-reference in our naïve theory of truth, in Section 3.

[7]There are numerous naïve theories of truth in the market that make use of a non-detachable conditional (e.g. [25], [3]). Some of them are paraconsistent, and some are not. In turn, paraconsistent logics have received many definitions (see [25, 3]). Perhaps the most encompassing definition describes a paraconsistent logic as a formal system where the inference from $A, \neg A$ to B does not hold in general, i.e. there is at least some instance of A and B where it fails.

[8]Paradigmatically, naïve theories of truth based on non-classical logics make use of the non-classical values to accommodate well-known paradoxical sentences –e.g in the context of paraconsistent theories of truth, paradoxical sentences are taken to be both true and false.

[9]Paraconsistent logics that have a consistency operator are nowadays called in the literature *Logics of Formal Inconsistency.* These systems go back to Newton da Costa's work on his C-systems in [9] and [8], but were later developed in a systematic way by W. Carnielli, J. Marcos, M. Coniglio and others in their seminal papers [6], [5].

[10]Other non-linguistic alternatives to recover classical logic in the context of non-classical logics are carried out in the meta-theory, e.g. Beall [2].

to fulfill additional desiderata related to achieving semantically closed theories. In this vein, if every semantic notion of the system should be expressible in a semantically closed theory, then surely every extensional semantic notion should be expressible in it. Thus, it seems that a semantically closed theory should be able to express every notion that corresponds to a function that can be displayed in a truth-table featuring only the truth values of the theory. Given this, we consider that a fourth desideratum for a naïve theory of truth considered as a first step towards a semantically closed theory, is for it achieve this sort of expressive completeness.[11] There is, of course, a close but non-trivial relationship between this desideratum and that of a logic being functionally complete, and indeed we will be commenting on this relationship below.

In summary, this paper will present a non-trivial naïve theory of truth that we will call **MSC+** (for "Matrix for Semantic Closure"), that satisfies the four desiderata specified above. The paper is structured as follows. In Section 2 we give some general formal and technical details about the systems we are going to work with. In Section 3 we prove some limitative results on various paraconsistent logics that might, in principle, be able to fulfill the four desiderata. In Section 4 we present the particular base logic we will use: **MSC**. In Section 5 we sketch our general strategy to achieve a theory that does indeed satisfy our four desiderata, which will require a particular way of expressing self-reference. In Section 6 we present a sequent system **LSC+** that is sound and complete with respect to **MSC+** and show that the presented theory has some meta-logic properties related to functional completeness. Finally, in Section 7 we make some concluding remarks and point out some further issues that might be interesting for future research.

2. Preliminaries

In the sequel we are going to work, mainly, with logics considered from a semantic point of view. We denote logical systems $\mathbf{L}$ with pairs $\langle \mathcal{L}, \vDash_{\mathbf{L}} \rangle$ of an uninterpreted language $\mathcal{L}$ and a (semantic) consequence relation $\vDash_{\mathbf{L}}$ induced, in turn, by a semantic structure (a *matrix*) $\mathcal{M}_{\mathbf{L}}$, intended to interpret the language $\mathcal{L}$.

In each case, we will work with a base language $\mathcal{L}$ composed of a denumerable set of propositional variables Var and a set C of n-ary connectives. The set of formulae $\mathsf{Form}_{\mathcal{L}}$ is defined, standardly, as the absolutely free algebra generated by Var over C.

Now, since we want to deal with a truth predicate and this linguistic possibility often implies the rise of potentially pathological phenomena represented by e.g. self-referential sentences that lead to paradoxes, we will need formal devices to model these facts.

In particular, we will need a symbol Tr for a (unary) truth predicate. Moreover, we will need *some* way to talk about the expressions of the language in the language itself by, e.g. having names to refer to the sentences of the language. We will do this, by means of a name-forming device $\langle \cdot \rangle$, which allows to add to the language a term $\langle A \rangle$, i.e. a name for A, for each formula A of the language. We will refer to the extended language that has all these resources as $\mathcal{L}^+$.

[11]Therefore, being expressively complete is a *necessary condition* for a theory to be semantically closed. Of course, we do not pretend that it is also a sufficient condition, because there might be semantic notions that are intensional, or hyperintensional, that cannot be captured by any truth-function. Thus, even a expressively complete theory may not be able express them.

This being said, we will denote formulae of $\mathcal{L}$ (either sentences –e.g. closed formulae– or open formulae) by capital Roman letters A, B, C, etc., and set of formulae of the language by capital Greek letters Γ, Δ, Σ, etc.

Definition 2.1 (Semantic Structure)**.** A semantic structure (a *matrix*) $\mathcal{M}_{\mathbf{L}}$ for the language $\mathcal{L}$ is a tuple $\mathcal{M}_{\mathbf{L}} = \langle \mathcal{V}_{\mathbf{L}}, \mathcal{D}_{\mathbf{L}}, \mathcal{O}_{\mathbf{L}} \rangle$ where

- $\mathcal{V}_{\mathbf{L}}$ is a (non-empty) set of truth values
- $\mathcal{D}_{\mathbf{L}}$ is a (non-empty) proper subset of $\mathcal{V}_{\mathbf{L}}$
- $\mathcal{O}_{\mathbf{L}}$ is a set that includes for every n-ary operator $\diamond$, the corresponding n-ary truth-function $\widetilde{\diamond} : (\mathcal{V}_{\mathbf{L}})^n \longrightarrow \mathcal{V}_{\mathbf{L}}$

All n-ary truth-functions belonging to each $\mathcal{O}_{\mathbf{L}}$ will be displayed in truth-tables. For the sake of clarity, when some connective $\diamond$ has a different associated truth-function in the structures $\mathcal{M}_{\mathbf{L}^a}$ and $\mathcal{M}_{\mathbf{L}^b}$ characteristic of the logics $\mathbf{L}^a$ and $\mathbf{L}^b$, we will call this functions, respectively, $\widetilde{\diamond}_{\mathbf{L}^a}$ and $\widetilde{\diamond}_{\mathbf{L}^b}$.

Definition 2.2 (Valuation)**.** Let $\mathcal{M}_{\mathbf{L}} = \langle \mathcal{V}_{\mathbf{L}}, \mathcal{D}_{\mathbf{L}}, \mathcal{O}_{\mathbf{L}} \rangle$ be a semantic structure (a matrix) for a language $\mathcal{L}$. An $\mathcal{M}_{\mathbf{L}}$-*valuation* for $\mathcal{L}$ is a homomorphism $v : \mathsf{Form}_{\mathcal{L}} \longrightarrow \mathcal{V}_{\mathbf{L}}$. We denote the set of all such valuations as $\mathsf{V}_{\mathbf{L}}$.

Moreover, below we will also require from the semantic structures interpreting these languages, to give a *transparent* interpretation to Tr, e.g. to ensure that if A is a sentence and $\langle A \rangle$ is a name for such sentence, then, for any valuation $v \in \mathsf{V}_{\mathbf{L}}$, $v(A) = v(Tr\langle A \rangle)$.

Definition 2.3 (Semantic Consequence Relation)**.** Given a semantic structure (a matrix) $\mathcal{M}_{\mathbf{L}}$, the consequence relation $\vDash_{\mathbf{L}}$ induced by $\mathcal{M}_{\mathbf{L}}$ is defined in the following way. We say that a formula A is a *semantic consequence* of a set of formulae Γ (notation $\Gamma \vDash_{\mathbf{L}} A$) if and only if for every valuation $v \in \mathsf{V}_{\mathbf{L}}$, if $v(B) \in \mathcal{D}_{\mathbf{L}}$ for every $B \in \Gamma$, then $v(A) \in \mathcal{D}_{\mathbf{L}}$.

Finally, we must provide some detail to the extent of how is self-reference achieved in the theory in question. There seem to be two main options to represent self-reference: through a *weak* or a *strong* procedure. The former option achieves this goal by requiring a self-referential sentence to be *equivalent* to a sentence that "talks about" the first one. The latter involves an essential use of identities.

These options might be instantiated by a plethora of technical means, varying from one framework to the other. In this paper, we will discuss some examples thereof.

Definition 2.4 (Weak Self-Referential Procedure)**.** Let **Th** be a theory that has a name-forming device $\langle \cdot \rangle$. If for every formula $A(x)$, with x as the only free variable in $A(x)$, there is a (closed) formula B such that the formula $B \leftrightarrow A(\langle B \rangle)$ is true in the context of **Th**, then we say that **Th** adopts a *weak* self-referential procedure.

Definition 2.5 (Strong Self-Referential Procedure)**.** Let **Th** be a theory that has a name-forming device $\langle \cdot \rangle$. If for every formula $A(x)$, with x as the only free variable, there is a term t such that t is identical to the name of $A(t)$ in the context of **Th**, then we say that **Th** adopts a *strong* self-referential procedure.

Remark 2.6. Notice that these procedures can sometimes be already present "in" the theories in question, e.g. i.e. if they are extensions of arithmetical theories or expressively equivalent theories that validate the Weak and/or the Strong Diagonal

Lemma. But in some cases, they do not come with the theories, in which case they can be "imposed from the meta-language" to the theories in question, e.g. by restricting their valuations.

In fact, the latter is the route followed, e.g. by David Ripley in [26] to adopt a strong self-referential procedure, in the context of a theory that does not necessarily extend an arithmetic system or anything of the like. We should emphasize that we will be following a similar route, but to adopt a weak self-referential procedure, again, in the context of a theory that does not necessarily extend an arithmetic system or anything of the like.

We will take the notational liberty of calling a theory **Th** that uses a Weak Self-Referential procedure, alternatively, as $\mathbf{Th_w}$ and –analogously– a theory **Th** that uses a Strong Self-Referential procedure, alternatively, as $\mathbf{Th_s}$. When context provides enough clarification, we will allow ourselves to omit these subscripts.

3. How not to build a naïve theory of truth

Let us briefly remember what we are after in this essay. We want a non-trivial paraconsistent naïve theory of truth that is also capable of recovering classical reasoning, and that is functionally complete. In this section we will analyze the case of some paraconsistent logics already presented in the literature to determine if they can, or cannot, accomplish these goals. To do this, we will first check whether or not these systems can be extended to naïve theories of truth that are capable of recovering classical reasoning, leaving functional completeness aside for now. After all, if the resulting theories were to fail at this stage of the investigation, that will be sufficiently informative for our purposes.

Thus, in this section we will be considering naïve theories of truth built on top of paraconsistent logics that are capable of recovering classical reasoning by means dear to Logics of Formal Inconsistency (LFIs, for short). In a nutshell, when $\circ(A)$ is a formula depending exactly on A, LFIs are systems for which the following holds (see e.g. [5], [4], [6])

There are some some Γ, A, B such that:

- $\Gamma, A, \neg A \nvDash B$
- $\Gamma, \circ(A), A \nvDash B$
- $\Gamma, \circ(A), \neg A \nvDash B$

And yet for all Γ, A, B:

- $\Gamma, \circ(A), A, \neg A \vDash B$

Along these lines, when the sets $\circ(A)$ is a singleton it is usually called a *consistency* operator, and is symbolized as $\circ A$.

Hence, in what follows we will study, first, the case of a naïve theory of truth built on top of the system **MPT** (developed in [7]), due to Marcelo Coniglio and Luis Silvestrini. Secondly, we will study the case of a naïve theory of truth built on top of the system $\mathbf{LP}^{\circ}$, i.e. an LFI that extends with a consistency operator the most popular paraconsistent logic in the literature about theories of truth: Graham Priest's **LP**.[12] In principle, these systems present major advantages towards our final goal. We will show, nevertheless, that under some circumstances these theories are trivial.

[12]For more about **LP**, see e.g. [23].

Let us consider, then, if **MPT**, our first candidate, may be the base of an adequate naïve theory of truth.

Definition 3.1. Let the language $\mathcal{L}_{\mathbf{MPT}}$ be composed of a denumerable set of propositional variables Var and a set $\mathsf{C}_{\mathbf{MPT}} = \{\sim, \neg, \wedge, \vee, \circ, \rightarrow_{\mathbf{MPT}}, \leftrightarrow_{\mathbf{MPT}}\}$ of connectives. The set of formulae $\mathsf{Form}_{\mathcal{L}_{\mathbf{MPT}}}$ is defined, standardly, as the absolutely free algebra generated by Var over $\mathsf{C}_{\mathbf{MPT}}$.

It is worth noticing that $\leftrightarrow_{\mathbf{MPT}}$ is just the conjunction of the left-right and the right-left conditionals, as in classical logic.

Definition 3.2. The logic $\mathbf{MPT} = \langle \mathcal{L}, \vDash_{\mathbf{MPT}} \rangle$ is defined by the semantic structure $\mathcal{M}_{\mathbf{MPT}} = \langle \mathcal{V}_{\mathbf{MPT}}, \mathcal{D}_{\mathbf{MPT}}, \mathcal{O}_{\mathbf{MPT}} \rangle$, built in the following way:

- $\mathcal{V}_{\mathbf{MPT}} = \{1, \frac{1}{2}, 0\}$
- $\mathcal{D}_{\mathbf{MPT}} = \{1, \frac{1}{2}\}$
- $\mathcal{O}_{\mathbf{MPT}}$ = the set of truth-functions associated with the connectives in $\mathsf{C}_{\mathbf{MPT}}$, displayed in the truth-tables of Figure 1

$\neg A$	A
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

$\sim A$	A
0	1
0	$\frac{1}{2}$
1	0

$\circ A$	A
1	1
0	$\frac{1}{2}$
1	0

$A \wedge B$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$A \vee B$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$A \rightarrow_{\mathbf{MPT}} B$	1	$\frac{1}{2}$	0
1	1	1	0
$\frac{1}{2}$	1	1	0
0	1	1	1

$A \leftrightarrow_{\mathbf{MPT}} B$	1	$\frac{1}{2}$	0
1	1	1	0
$\frac{1}{2}$	1	1	0
0	0	0	1

FIGURE 1. Truth-tables for the logic **MPT**

Definition 3.3. Let $\mathbf{MPT_s}+$ be the theory formulated in the language $\mathcal{L}^{+}_{\mathbf{MPT}}$ (an extension of $\mathcal{L}_{\mathbf{MPT}}$, as described in Section 2), that is the result of:

- adopting a *strong* self-referential procedure
- restricting **MPT**+'s valuations to those that satisfy Transparency

Is $\mathbf{MPT_s}+$ an admissible naïve theory of truth? We are going to show that the answer is negative.

Theorem 3.4. $\mathbf{MPT_s}+$ *is unsatisfiable.*

Proof. Consider a name A, such that $A = \langle (Tr(A) \rightarrow \bot) \rangle$,[13] which can be considered as a Curry sentence. We will show that there is no stable assignment of truth values to $Tr(A)$. If $v(Tr(A)) = 0(1)$, then $v(Tr(A) \rightarrow \bot) = 1(0)$. Now, if $v(Tr(A)) = \frac{1}{2}$, then $v(Tr(A) \rightarrow \bot) = 0$. Thus, as **MPT**+'s truth predicate is transparent, and $A = \langle (Tr(A) \rightarrow \bot \rangle$, then $v(Tr(A)) = 0$. But the initial assumption was that $v(Tr(A)) = \frac{1}{2}$. Thus, there is no stable assignment of truth values to this sentence.[14] □

[13] Notice that $\bot$ can be defined in **MPT**+ as $\neg(B \rightarrow B)$, for any sentence B. Another way to do it is as $B \wedge \sim B$. But as the authors define the classical negation in terms of the conditional, the former seems a more basic one.

[14] If **MPT**+ uses a metalinguistic function f from names to sentences such that, for every formula $A(x)$, with x as the only free variable, there is a term t such that $f(t) = A(t)$, and we only consider

It is interesting to highlight that **MPT**'s conditional –which was centrally involved in the unsatisfiability result above– has a distinctive classical feature: it validates *Modus Ponens*. In the literature about truth theories, it is customary for paraconsistent systems to avoid conditionals that validate *Modus Ponens* –precisely in light of paradoxes such as Curry's. For instance, **LP**'s conditional invalidates this rule. Thus, it is argued, a theory of truth based on **LP** will not have problems with a Curry sentence, because there is a stable valuation for it: A may just get the value $\frac{1}{2}$. If this is the case for every pathological sentence that can be built, then maybe **LP** can be used as the logical basis for the kind of semantically closed theory we are looking for.

Let us consider, then, if **LP**+, our second candidate, may be the base of an adequate naïve theory. It is well known that it is possible to add a transparent truth predicate to **LP**, obtaining the non-trivial theory of truth **LP**+.[15] But in order for it to cope with our desiderata of recovering classical reasoning in consistent contexts, and to do that *in the language*, **LP**'s language has to be expanded.[16] Let us call the base logic that results from expanding **LP** with a consistency operator, **LP**°.

Definition 3.5. Let the language $\mathcal{L}_{\mathbf{LP}^\circ}$ be composed of a denumerable set of propositional variables Var and a set $\mathsf{C}_{\mathbf{LP}^\circ} = \{\neg, \wedge, \vee, \circ, \rightarrow_{\mathbf{LP}}, \leftrightarrow_{\mathbf{LP}}\}$ of connectives. The set of formulae $\mathsf{Form}_{\mathcal{L}_{\mathbf{LP}^\circ}}$ is defined, standardly, as the absolutely free algebra generated by Var over $\mathsf{C}_{\mathbf{LP}^\circ}$.

Again, it is worth noticing that $\leftrightarrow_{\mathbf{LP}}$ is just the conjunction of the left-right and the right-left conditionals, as in classical logic.

Definition 3.6. The logic $\mathbf{LP}^\circ = \langle \mathcal{L}, \vDash_{\mathbf{LP}^\circ} \rangle$ is defined by a the semantic structure $\mathcal{M}_{\mathbf{LP}^\circ} = \langle \mathcal{V}_{\mathbf{LP}^\circ}, \mathcal{D}_{\mathbf{LP}^\circ}, \mathcal{O}_{\mathbf{LP}^\circ} \rangle$, which is obtained in the following way:

- $\mathcal{V}_{\mathbf{LP}^\circ} = \{1, \frac{1}{2}, 0\}$
- $\mathcal{D}_{\mathbf{LP}^\circ} = \{1, \frac{1}{2}\}$
- $\mathcal{O}_{\mathbf{LP}^\circ}$ = the set of truth-functions associated with the connectives in $\mathsf{C}_{\mathbf{LP}^\circ}$, displayed in the truth-tables of Figure 2

$A \rightarrow_{\mathbf{LP}} B$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1	1

$A \leftrightarrow_{\mathbf{LP}} B$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	0	$\frac{1}{2}$	1

FIGURE 2. Truth-tables for the logic **LP**°

valuations where t refers to $A(t)$, there will be a term u such that $f(u) = (Tr(u) \rightarrow \perp)$. Then there is no stable assignment of truth values to $Tr(u)$. If $v(Tr(u)) = 0(1)$, then $v(Tr(u) \rightarrow \perp) = 1(0)$. Now, if $v(Tr(u)) = \frac{1}{2}$, then $v(Tr(u) \rightarrow \perp) = 0$. Thus, as **MPT**+'s truth predicate is transparent, and $f(u) = (Tr(u) \rightarrow \perp)$, then $v(Tr(u)) = 0$. But the initial assumption was that $v(Tr(u)) = \frac{1}{2}$. Thus, there is no stable assignment of truth values to this sentence.

[15] For more about such results, see [23], [24] and [25]

[16] Of course, other available options have been explored. Priest, in [25], and Beall, in [3], have presented other ways to do it. But Priest's option specified this restriction from the metalanguage. Beall, on the other hand, uses multiple conclusion logics.

It should be highlighted that $\rightarrow_{\mathbf{LP}}$ is the definable material conditional, which can be formulated as the disjunction of the negation of the antecedent and the assertion of the consequent. The remaining connectives for which we did not provide new truth-tables, maintain their meanings assigned in $\mathcal{M}_{\mathbf{MPT}}$.

Definition 3.7. Let $\mathbf{LP}^{\circ}_{\mathbf{s}}+$ be the theory formulated in the language $\mathcal{L}^{+}_{\mathbf{LP}^{\circ}}$ (an extension of $\mathcal{L}_{\mathbf{LP}^{\circ}}$, as described in Section 2), that is the result of:

- adopting a *strong* self-referential procedure
- restricting $\mathbf{LP}^{\circ}+$'s valuations to those that satisfy Transparency

Is $\mathbf{LP}^{\circ}_{\mathbf{s}}+$ and adequate naïve theory of truth, that respects all of the above desiderata? Once again, the answer is negative. To prove this fact, we need a few preliminary results.

Theorem 3.8. $\mathbf{LP}^{\circ}_{\mathbf{s}}+$ *is unsatisfiable.*

Definition 3.9 (Classical Negation)**.** Let a logic $\mathbf{L}$ be a tuple $\langle \mathcal{L}, \vDash_{\mathcal{M}_{\mathbf{L}}} \rangle$ and let $\mathsf{V}_{\mathcal{M}_{\mathbf{L}}}$ be the set of its valuations. We say (following some remarks made by De and Omori in [10]), that a unary operator $\rightharpoondown$ is a *classical negation* for the logic $\mathbf{L}$ if and only if, for every sentence A and for every valuation $v \in \mathsf{V}_{\mathbf{L}}$:

$$v(A) \in \mathcal{D}_{\mathbf{L}} \text{ if and only if } v(\rightharpoondown A) \notin \mathcal{D}_{\mathbf{L}}$$

Remark 3.10. A classical negation is definable in $\mathbf{LP}^{\circ}+$ as $\rightharpoondown A =_{def} \neg A \wedge \circ A$.

Lemma 3.11. *If a (many-valued) naïve theory of truth* **Th** *is capable of defining a classical negation, and also uses a strong self-referential procedure, then* **Th** *is unsatisfiable.*

Proof. Let C be a name such that $C = \langle \rightharpoondown Tr(C) \rangle$. Then, there would be no stable assignment of truth values to $Tr(C)$. We just need to consider two possibilities: (i) either C receives a designated valued, or (ii) it receives an undesignated value. As **Th** has a transparent truth predicate, (i) if $v(Tr(C)) \in \mathcal{D}$, then $v(\rightharpoondown Tr(C)) \notin \mathcal{D}$. And (ii) if $v(Tr(C)) \notin \mathcal{D}$, then $v(\rightharpoondown Tr(C)) \in \mathcal{D}$.[17] □

Therefore, as a corollary, since $\mathbf{LP}^{\circ}_{\mathbf{s}}+$ is able to define a classical negation, has a transparent truth predicate and uses a strong self-referential procedure.[18]

Finally, if we want to avoid this kind of results there seems to be two immediate options:

(1) *either* we give up some expressive requirements (such as having consistency or recovery operators, classical negations, a transparent truth predicate, etc.)
(2) *or* we give up the need for a *strong* self-referential procedure

[17]Or, if the theory uses a metalinguistic function f from names to sentences such that, for every formula $A(x)$, with x as the only free variable, there is a term t such that $f(t) = A(t)$, and we only consider models where t refers to $A(t)$, there will be a term u such that $f(s) = \rightharpoondown Tr(s)$. Then there is no stable assignment of truth values to $Tr(s)$. We just need to consider two possibilities: (i) either $Tr(s)$ receives a designated valued, or (ii) it receives an undesignated value. As the theory has a transparent truth predicate, (i) if $v(Tr(s)) \in \mathcal{D}$, then $v(\rightharpoondown Tr(s)) \notin \mathcal{D}$. And (ii) if $v(Tr(s)) \notin \mathcal{D}$, then v$(\rightharpoondown Tr(s)) \in \mathcal{D}$.

[18]Notice that as **MPT**+ is able to define a classical negation in exactly the same way as $\mathbf{LP}^{\circ}+$, Theorem 3.4 could be seen as a corollary of Lemma 3.11. Thanks to an anonymous referee for this clarification.

The first path seems to be easier. But notice that if we take it, we cannot fulfill our goal of having a semantically closed system. If we do not want to give up the project of having a semantically closed theory, the only admissible option seems to be the last one.

In the rest of the paper, we will explore the prospect of changing the way to achieve self-reference, using some *weak* self-referential procedure. This will be done in the context of a new system, that will turn out to be sufficiently expressive under the eyes of our desiderata. To do this, a new conditional will play a key part in this plot.

4. A logic for semantic closure

We present below our preferred base logic called **MSC**, for "Matrix Logic for Semantic Closure".

Definition 4.1. Let the language $\mathcal{L}_{\mathbf{MSC}}$ be composed of a denumerable set of propositional variables Var and a set $\mathsf{C}_{\mathbf{MSC}} = \{\neg, \wedge, \vee, \circ, \rightarrow_{\mathbf{LP}}, \leftrightarrow_{\mathbf{LP}}\}$ of connectives. The set of formulae $\mathsf{Form}_{\mathcal{L}_{\mathbf{MSC}}}$ is defined, standardly, as the absolutely free algebra generated by Var over $\mathsf{C}_{\mathbf{MSC}}$.

Definition 4.2. The logic $\mathbf{MSC} = \langle \mathcal{L}_{\mathbf{MSC}}, \vDash_{\mathbf{MSC}} \rangle$ is defined by a the semantic structure $\mathcal{M}_{\mathbf{MSC}} = \langle \mathcal{V}_{\mathbf{MSC}}, \mathcal{D}_{\mathbf{MSC}}, \mathcal{O}_{\mathbf{MSC}} \rangle$, which is obtained in the following way:

- $\mathcal{V}_{\mathbf{MSC}} = \{1, \frac{1}{2}, 0\}$
- $\mathcal{D}_{\mathbf{MSC}} = \{1, \frac{1}{2}\}$
- $\mathcal{O}_{\mathbf{MSC}}$ = the set of truth-functions associated with the connectives in $\mathsf{C}_{\mathbf{MSC}}$, that are displayed in the truth-tables of Figure 3

$A \rightarrow_{\mathbf{MSC}} B$	1	$\frac{1}{2}$	0
1	1	1	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

$A \leftrightarrow_{\mathbf{MSC}} B$	1	$\frac{1}{2}$	0
1	1	1	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	0	$\frac{1}{2}$	1

Figure 3. Truth-tables for the logic **MSC**

Again, it is worth noticing that $\leftrightarrow_{\mathbf{MSC}}$ is just the conjunction of the left-right and the right-left conditionals, as in classical logic. The remaining connectives for which we did not provide new truth-tables, maintain their meanings assigned in $\mathcal{M}_{\mathbf{MPT}}$.

As we will show in the next Section below, this distinctive conditional will help us design weak self-referential procedure that will be crucial to avoid triviality. Given our remarks in Section 7, many non-detachable conditionals coming from many-valued paraconsistent logics may be able to serve this role. However, even if we do not think of the presently discussed conditional as the only one that deserves attention to this regard, we are of the opinion that it has in fact some interesting philosophical properties –on which we comment below.

The first point we would like to highlight is that, using a slight modification of Tomova's terminology from [29], the conditional $A \rightarrow_{\mathbf{MSC}} B$ can be justifiably referred to as a *semi-natural* conditional given that, when restricted to classical values, it behaves classically.

Secondly, the conditional $A \rightarrow_{\mathbf{MSC}} B$ has an interesting connection with the external notion of *entailment*. It gets value 1 whenever a designated value is preserved from antecedent to consequent -either because the consequent is assigned the value $\frac{1}{2}$ or 1, or because the antecedent is assigned the value 0. In this sense, the conditional seems to partially internalize the notion of entailment of the theory: if a (single premise) inference is valid, then the corresponding conditional will get value 1. Unfortunately, this connection does not hold when we consider invalid inferences. In fact, there will be invalid inferences that do not correspond to a false conditional.

Thirdly, **MSC**'s conditional allows for an interesting conceptual reading. Let us start comparing **MSC**'s conditional with **LP**'s conditional. The latter can be shown to be such that $A \rightarrow_{\mathbf{LP}} B \dashv\vDash \neg A \vee B$. Thus, in **LP**, "if A, then B" is true if and only if either A is false, or B is true. Moreover, by known equivalences, $A \leftrightarrow_{\mathbf{LP}} B \dashv\vDash (A \wedge B) \vee (\neg A \wedge \neg B)$. Thus, in **LP**, "$A$ is equivalent to B" is true if and only if A and B belong to the same semantic category, i.e. they are both true or both false.

In the case of **MSC** the situation is relatively similar. Let $\triangle A$ mean, intuitively, that A gets a designated value or, more conceptually, that A is "true", in a metatheoretic sense.[19]. Let us notice that in the context of paraconsistent logics, while "being true" is an inconsistent notion (i.e. there are true formulae that are also not true), the notion of "being designated" is not inconsistent (i.e. there are not designated formulae that are also non-designated).

Moreover, it can be shown that $A \rightarrow_{\mathbf{MSC}} B \dashv\vDash \neg A \vee \triangle B$. Thus, in **MSC**, "if A, then B" is true if and only if either A is false, or B is designated. Additionally, by known equivalences, $A \leftrightarrow_{\mathbf{MSC}} B \dashv\vDash (\triangle A \wedge \triangle B) \vee (\neg A \wedge \neg B)$. Thus, in **MSC**, "$A$ is equivalent to B" is true if and only if A and B are both designated, or they are both false.

Conditionals		$\mathbf{LP}^{\circ}$+	**MPT+**	**MSC+**
Modus Ponens	$A \rightarrow B, A \vDash B$	No	Yes	No
Modus Ponens II	$(A \rightarrow B) \wedge A \vDash B$	No	Yes	No
Deduction Thm. (RTL)	$\vDash A \rightarrow B$, then $A \vDash B$	No	Yes	No
Deduction Thm. (LTR)	If $A \vDash B$, then $\vDash A \rightarrow B$	Yes	Yes	Yes
Identity	$\vDash A \rightarrow A$	Yes	Yes	Yes
Transitivity	$A \rightarrow B, B \rightarrow C \vDash A \rightarrow C$	No	Yes	No
Transitivity II	$(A \rightarrow B) \wedge (B \rightarrow C) \vDash A \rightarrow C$	No	Yes	No
Disjunction Intro.	$\vDash A \rightarrow (A \vee B)$	Yes	Yes	Yes
Conjunction Elim.	$\vDash (A \wedge B) \rightarrow A$	Yes	Yes	Yes
Contraposition	$\vDash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$	Yes	No	Yes
Contraction I	$A \rightarrow (A \rightarrow B) \vDash A \rightarrow B$	Yes	Yes	Yes
Contraction II	$\vDash A \rightarrow (A \rightarrow B) \rightarrow A \rightarrow B$	Yes	Yes	Yes
Pseudo MP I	$\vDash ((A \rightarrow B) \wedge A) \rightarrow B$	Yes	Yes	Yes
Pseudo MP II	$\vDash A \rightarrow ((A \rightarrow B) \rightarrow B)$	Yes	Yes	Yes
Transit. Liar	$\vDash ((\top \rightarrow \lambda) \rightarrow \bot$	Yes	No	No

FIGURE 4

[19]This notion is definable in **MSC** as follows $\triangle A =_{def} A \vee \neg \circ A$.

Finally, the only strange feature of $\rightarrow_{\mathbf{MSC}}$ is its main (and only) difference with **MPT**'s conditional: when the antecedent gets value $\frac{1}{2}$ and the consequent gets value 0, **MSC**'s conditional gets (the designated) value $\frac{1}{2}$. In fact, this is a feature shared with **LP**'s material conditional: if the conditional in question is read in a dialetheist way, it gets both from true to false –and thus it should be false– and from false to false –and hence should be true. Therefore, it should be (and it is) both true and false. Figure 4 shows a very brief comparison between **MSC**+'s, **MPT**+'s and **LP**°+'s conditional.

All this being said, let us now show how to achieve self-reference in **MSC**+, via a particular weak self-referential procedure.

5. Weak self-reference for **MSC**+

5.1. **Preliminary considerations.** The most simple way to adopt a weak self-referential procedure for a paraconsistent theory of truth would be to consider the basic paraconsistent theory of truth as the basis of a formal theory **Th** that has the same expressive power than, say, Peano Arithmetic has. More specifically, if the resulting theory *proves* the Weak Diagonal Lemma as a *theorem*, we would be granted the adoption of a procedure of this sort, as detailed in Section 2. However, it is convenient to highlight that the usual ways of doing this when the basic logic is paraconsistent are not so straightforward.

On the one hand, it is not so easy to show that the axioms of Peano Arithmetic together with an underlying paraconsistent theory of truth e.g. **LP**+ lead to a proof of the Weak Diagonal Lemma. It is also not easy to prove this for the case of **MSC**+. So considering an axiomatic arithmetic theory with an underlying paraconsistent theory of truth does not sound very promising.

On the other hand, it is not easy either to show that if we consider a sequent calculus for a paraconsistent theory of truth e.g. **LP**+ and add to it an arithmetic theory formulated with sequent rules (*à la* Negri and von Plato in [19], [18], applied for theories of truth in e.g. [12]), then this will lead to a proof of the Weak Diagonal Lemma. Of course, it is not easy to prove this for the case of **MSC**+. So considering an arithmetic theory formulated with sequent rules together with a sequent calculus for a paraconsistent theory of truth does not sound very promising either.

In conclusion, to consider the basic paraconsistent theory of truth as the basis of a theory **Th** that has the same expressive power than, say, Peano Arithmetic has is not a straightforward way to adopt a weak self-referential procedure for a paraconsistent theory of truth. This is why we need to validate this principle another way.

5.2. **Adopting a Weak Self-Referential Procedure.** Our way of doing this will be rather mechanical: we will build a set Z that will include all the instances of the desired weak self-referential procedure that contain appearances of the truth predicate Tr. By doing this, we will include in Z all variants of the usual (and unusual) pathological sentences, like *The Liar*, *Curry*, *The Truth-Teller*, etc, along with potentially new pathological expressions, like strengthened Liar and Curry sentences. Finally, we will show that the resulting theory $\mathbf{MSC_w}$+ is non-trivial.

In fact, we will show that there is a valuation that satisfies Z alongside with **MSC**+, though this valuation does not satisfy every formula.[20]. This valuation will be indeed very simple: it will assign the value $\frac{1}{2}$ to every propositional variable. As a corollary, this implies that the resulting theory adopts a weak self-referential procedure.

Now, to go into the technicalities of the construction of the set Z, let us consider the following. First, we will discriminate an infinite proper subset of propositional variables $\mathsf{Var}^* \subseteq \mathsf{Var}$. We will then mark –in the metalanguage– every member of Var^* with an *.[21] Thus, we will refer to the members of Var^* as, for example, p^*, q^*, etc.

Secondly, we will consider sentences of the form

$$x^* \leftrightarrow A_{x^*}$$

Where A_{x^*} refers to a sentence that has at least one instance of $Tr(\langle x^* \rangle)$ as a subformula; x^* is a member of Var^*; and the biconditional at play is the characteristic (non-detachable) biconditional of **MSC**.[22]

Instantiation on A_{x^*} and the biconditional statement will exemplify every self-referential sentence that includes a truth predicate to be represented in the language.[23] In particular, we will be able to model the traditional pathological sentences, but also possible new pathological sentences including the consistency operator or classical negation. Nevertheless, we need a further semantic restriction.

Finally, for each formula-schema A_{x^*} we will select one and only one corresponding biconditional with a different propositional letter as its left term. We will denote the set of all such biconditionals as Z. This set of biconditionals will be used as a surrogate of the set of potentially *pathological* sentences definable via e.g. the Weak Diagonal Lemma.

Definition 5.1. Let $\mathbf{MSC_w}$+ be the theory formulated in the language $\mathcal{L}^+_{\mathbf{MSC}}$ (an extension of $\mathcal{L}_{\mathbf{MSC}}$, as described in Section 2), that is the result of:

- adopting a *weak* self-referential procedure, i.e. by restricting **MSC**+'s valuations to those that satisfy all the biconditionals in Z
- restricting **MSC**+'s valuations to those that satisfy Transparency

To conclude this Section, we will now establish that in fact $\mathbf{MSC_w}$+ is non-trivial.

Theorem 5.2 (Non-Triviality)**.** *There is an* $\mathbf{MSC_w}$+ *valuation, for which not every formula is designated.*

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[20]This valuation does not necessarily provide an intended model for **MSC**+, but it does provide *a* model for it.

[21]Though this last move is not essential. But putting a mark to those distinguished propositional letters will make things easier to understand, as those propositional letters will play a key part in the self-referential procedure we are about to present.

[22]By that we just meant an instance of $Tr(y)$ where the y is substituted by the designated name of the relevant instance of x^*

[23]To be precise, not every pathological sentence can be built using the procedure just sketched. For example, it does not allow to build cycles –e.g. sentences that refer to other sentences that (eventually) refer to them. However, we prefer to leave these details out of the picture, for it will not be difficult (although it will be tedious) to expand this procedure to construct every pathological sentences. Still, as our main focus are not cycles (nor, for example, the sentences present in Yablo's paradox, see e.g. [30]) these remarks seem to us good enough, for the purpose of this paper.

Proof. Let us consider the valuation v that assigns to every propositional letter of Var (and hence of Var^*) the value $\frac{1}{2}$. Let p^* be the "regular" Liar, expressed with the paraconsistent negation, i.e. $p^* \leftrightarrow_{\mathbf{MSC}} \neg Tr_{p^*}$. Thus, $v(p^*) = \frac{1}{2}$. Now consider the consistency assertion relative to p^*, i.e. $\circ p^*$. If, $v(p^*) = \frac{1}{2}$, then $v(\circ p^*) = 0$. Thus, we have the desired formula which shows that $\mathbf{MSC_w}+$ is not trivial.

Then, we only need to check if v is an "admissible" valuation –i.e. if v assigns a designated value to every biconditional in Z receives a designated value. But, as we have established, every propositional letter q will be such that $v(q) = \frac{1}{2}$, including every propositional letter from Var^*. Thus, every biconditional $x^* \leftrightarrow A_{x^*}$ that belongs to Z will receive a designated value.[24]

□

Remark 5.3. Consider the *weakly self-referential* versions of the sentences used to prove the triviality of $\mathbf{MPT}+$ and $\mathbf{LP}^{\circ}+$, respectively: $s^* \leftrightarrow (Tr(\langle s^* \rangle) \to \bot$ and $u^* \leftrightarrow {\sim}Tr(\langle u^* \rangle)$. These sentences do not cause any problem in the context of $\mathbf{MSC_w}+$, for both s^* and u^* can be assigned the value $\frac{1}{2}$.

To conclude this section, let us remember that in [13] Laura Goodship suggested two general recommendations for naïve theories, which included naïve set-theory, naïve theories of truth, etc. She stressed that if we want the theory to be safe from trivialization due to semantic paradoxes and at the same time to retain a huge expressive power, there seems to be two main modifications that can be done.

One of the last things that we would need to establish, in order to verify that we accomplished the aim of this paper is whether or not the T-Scheme holds unrestrictedly. We conclude that this is indeed the case, a result of the following theorem.

Theorem 5.4. *The T-Scheme is unrestrictedly valid in* $\mathbf{MSC_w}+$

Proof. Notice that every $\mathbf{MSC_w}+$ valuation satisfies Transparency, i.e. every valuation v is such that, for each sentence A, $v(A) = v(Tr(\langle A \rangle))$. Moreover, notice that the truth-table for the characteristic biconditional of $\mathbf{MSC_w}+$ is such that if for every valuation v, $v(B) = v(C)$, then $\vDash_{\mathbf{MSC_w}+} B \leftrightarrow C$. Therefore, as an instance, $\vDash_{\mathbf{MSC_w}+} A \leftrightarrow Tr(\langle A \rangle)$. In other words, the T-Scheme is unrestrictedly valid in $\mathbf{MSC_w}+$. □

These modifications regarded the (bi)conditional which is used in the formulation of the comprehension axioms: (1) either the conditional should invalidate *Modus Ponens*, or (2) the conditional should invalidate *Contraction* and *Pseudo Modus Ponens*. Naïve theories embracing the first option are referred to in the literature as being part of the "Goodship Project" (see [2]). This project was already proven fruitful in the context of naïve set-theories based on paraconsistent logics in Hitoshi Omori's paper [20].

In this vein, the present paper constitutes the first successful exploration of the Goodship Project in the context of naïve theories of truth based on paraconsistent logics: we presented a naïve theory of truth that uses a conditional lacking *Modus Ponens* to express self-reference.

[24]We would like to thank Dave Ripley for the help provided with regard to this result.

6. Metalogical properties: Soundness, Completeness and Expressive-Completeness

6.1. **Soundness, Completeness, Non-Triviality.** We will use, as the target proof theory, the three-sided disjunctive sequent system **LSC+**. We now specify how disjunctive sequents behave.

Definition 6.1. A disjunctive sequent $\Gamma \mid \Sigma \mid \Delta$ is satisfied by a valuation v if and only if $v(\gamma) = 0$ for some $\gamma \in \Gamma$, or $v(\sigma) = \frac{1}{2}$ for some $\sigma \in \Sigma$, or $v(\delta) = 1$ for some $\delta \in \Delta$. A sequent is valid if and only if it is satisfied by every valuation. A valuation is a counterexample to a sequent if the valuation does not satisfy the sequent.[25]

There is a strong relation between valid sequents of **LSC+** and valid inferences of $\mathbf{MSC_w}$+, i.e. $\Gamma \vDash_{\mathbf{MSC_w}+} \Delta$ if and only if $\Gamma \mid \Delta \mid \Delta$ is valid. This fact follows from the definition of $\mathbf{MSC_w}$+'s validity and the definition of validity of a three-sided sequent.

The proof system we are about to present includes some axioms, alongside with structural and operational rules. As is usual, a sequent is *provable* in **LSC+** if and only if it follows from the axioms by some number (possibly zero) of applications of the rules.

Given we are working with sets, the effects of the structural rules of Exchange and Contraction are built in, and Weakening is built into the axioms. Still, to make things easy to understand, we will include a Structural Rule of Weakening. We will have three versions of a three-sided Cut rule, and also a Derived Cut rule (that can be inferred from the three basic Cut rules) and that will be a key part of the Completeness Proof. Id and SeudoDL (short for "Seudo Diagonal Lemma") are the axiom-schemes of **LSC+**. Weak, Cut1, Cut2, Cut3 and Derived Cut are structural rules. The rest of the rules presented below are the operational rules of **LSC+**. Notice that, in order to apply the rule SeudoDL, the proviso that $x^* \leftrightarrow A_{x^*}$ is a member of Z must be satisfied.

- **Axioms:**

$$\text{Id}\ \frac{}{A, \Gamma \mid A, \Sigma \mid A, \Delta} \qquad \text{SeudoDL}\ \frac{}{\Gamma \mid x^* \leftrightarrow A_{x^*}, \Sigma \mid x^* \leftrightarrow A_{x^*}, \Delta}$$

- **Structural Rules:**

$$\text{Weak}\ \frac{\Gamma \mid \Sigma \mid \Delta}{\Gamma, \Gamma' \mid \Sigma, \Sigma' \mid \Delta, \Delta'} \qquad \text{Cut 2}\ \frac{\Gamma \mid \Sigma \mid \Delta, A \qquad \Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma \mid \Delta}$$

$$\text{Cut 1}\ \frac{\Gamma, A \mid \Sigma \mid \Delta \qquad \Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma \mid \Delta} \qquad \text{Cut 3}\ \frac{\Gamma, A \mid \Sigma \mid \Delta \qquad \Gamma \mid \Sigma \mid \Delta, A}{\Gamma \mid \Sigma \mid \Delta}$$

$$\text{Derived Cut}\ \frac{\Gamma, A \mid \Sigma, A \mid \Delta \qquad \Gamma \mid \Sigma, A \mid \Delta, A \qquad \Gamma, A \mid \Sigma \mid \Delta, A}{\Gamma \mid \Sigma \mid \Delta}$$

[25] It might be argued that inferences involve formulae (and not sets of them) as conclusions. In this case, the conclusion should be read as a single formula, or the singleton of a single formula. All the results below carry over to this approach without loss of generality.

- **Operational Rules:**

$$\text{L}\neg\ \frac{\Gamma \mid \Sigma \mid \Delta, A}{\Gamma, \neg A \mid \Sigma \mid \Delta} \qquad \text{M}\neg\ \frac{\Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma, \neg A \mid \Delta} \qquad \text{R}\neg\ \frac{\Gamma, A \mid \Sigma \mid \Delta}{\Gamma \mid \Sigma \mid \Delta, \neg A}$$

$$\text{L}\wedge\ \frac{\Gamma, A, B \mid \Sigma \mid \Delta}{\Gamma, A \wedge B \mid \Sigma \mid \Delta} \quad \text{M}\wedge\ \frac{\Gamma \mid \Sigma, A \mid \Delta, A \qquad \Gamma \mid \Sigma, B \mid \Delta, B \qquad \Gamma \mid \Sigma, A, B \mid \Delta}{\Gamma \mid \Sigma, A \wedge B \mid \Delta}$$

$$\text{R}\wedge\ \frac{\Gamma \mid \Sigma \mid \Delta, A \qquad \Gamma \mid \Sigma \mid \Delta, B}{\Gamma \mid \Sigma \mid \Delta, A \wedge B}$$

$$\text{L}{\to}\ \frac{\Gamma \mid \Sigma \mid \Delta, A \qquad \Gamma, B \mid \Sigma \mid \Delta}{\Gamma, A \to B \mid \Sigma \mid \Delta} \qquad \text{M}{\to}\ \frac{\Gamma \mid \Sigma, A \mid \Delta \qquad \Gamma, B \mid \Sigma \mid \Delta}{\Gamma \mid \Sigma, A \to B \mid \Delta}$$

$$\text{R}{\to}\ \frac{\Gamma, A \mid \Sigma, B \mid \Delta, B}{\Gamma \mid \Sigma \mid \Delta, A \to B}$$

$$\text{L}\circ\ \frac{\Gamma \mid \Sigma, A \mid \Delta}{\Gamma, \circ A \mid \Sigma \mid \Delta} \quad \text{R}\circ\ \frac{\Gamma, A \mid \Sigma \mid \Delta, A}{\Gamma \mid \Sigma \mid \Delta, \circ A} \qquad \text{L}{\sim}\ \frac{\Gamma \mid \Sigma, A \mid \Delta, A}{\Gamma, {\sim} A \mid \Sigma \mid \Delta} \quad \text{R}{\sim}\ \frac{\Gamma, A \mid \Sigma \mid \Delta}{\Gamma \mid \Sigma \mid \Delta, {\sim} A}$$

$$\text{L}Tr\ \frac{\Gamma, A \mid \Sigma \mid \Delta}{\Gamma, Tr(\langle A \rangle) \mid \Sigma \mid \Delta} \qquad \text{M}Tr\ \frac{\Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma, Tr(\langle A \rangle) \mid \Delta} \qquad \text{R}Tr\ \frac{\Gamma \mid \Sigma \mid \Delta, A}{\Gamma \mid \Sigma \mid \Delta, Tr(\langle A \rangle)}$$

As the rest of the constants ($\vee$ and $\leftrightarrow_{\mathbf{MSC}}$) can be defined in terms of the former, and hence we will not specify rules for them.

The following are some important properties of **MSC**+ and **LSC**+:

Theorem 6.2 (Soundness). *If a sequent* $\Gamma \mid \Sigma \mid \Delta$ *is provable, then it is valid.*

Proof. The axioms are valid, and validity is preserved by the rules, as can be checked without too much trouble.

□

Theorem 6.3 (Completeness). *If a sequent* $\Gamma \mid \Sigma \mid \Delta$ *is valid, then it is provable.*

Proof. In the Appendix. □

Theorem 6.4 (Non-Triviality). *There is at least one sequent* $\Gamma \mid \Sigma \mid \Delta$ *that is not derivable in* **LSC**+

Proof. Let p^* be The Liar sentence. Given the sequent $\varnothing \mid p^* \mid \varnothing$ is valid and the system is complete, the sequent $\varnothing \mid p^* \mid \varnothing$ will be provable in the **LSC**+. Moreover, given the sequent $p^* \mid \varnothing \mid \varnothing$ is not valid and **LSC**+ is sound, $p^* \mid \varnothing \mid \varnothing$ will not have a proof.

□

6.2. **Remarks on functional completeness and related notions.** Below we present the definitions of functional completeness and truth completeness due to Malinowski, Rosser and Turquette with some slight modifications to cope with our notation.

Definition 6.5 ([17])**.** For any natural number $n \geq 2$, we denote by U_n any algebra of the form $U_n = \langle \mathcal{V}_n, \mathcal{O}_n \rangle$ where $|\mathcal{V}_n| = n \in \mathbb{N}$ and $\mathcal{O}_n$ is a set of finitary operations on $\mathcal{V}_n$. Then,

- g *defines* f in $\mathcal{O}_n$ if g is a composition of some of $\mathcal{O}_n$ such that: $f(\vec{x}) = g(\vec{x})$ for all $\vec{x} \in E_n^k$
- f is *definable* in $\mathcal{O}_n$ if g defines f in $\mathcal{O}_n$ for some g
- U_n is *functionally complete* if every finitary k-ary ($k \in \mathbb{N}$) mapping $f : E_n^k \longrightarrow \mathcal{V}_n$ is definable in $\mathcal{O}_n$
- (Rosser and Turquette [27]) U_n is *truth complete* if all J-operators $J_i : i \in \mathcal{V}_n$ are definable in U_n, where $J_i(x) = 1$ if $x = i$, and $J_i(x) = 0$ otherwise

Remark 6.6. Although the above definition talks of algebras being functionally complete or truth complete, it is standard to speak of logics themselves enjoying these features. This is, of course, a reasonable thing to do since n-valued logics $\mathbf{L} = \langle \mathcal{L}, \vDash_{\mathbf{L}} \rangle$ have a consequence relation $\vDash_{\mathbf{L}}$ induced by a matrix $\mathcal{M}_{\mathbf{L}} = \langle \mathcal{V}_n, \mathcal{D}_n, \mathcal{O}_n \rangle$ with regard to this which, $\langle \mathcal{V}_n, \mathcal{O}_n \rangle$ is an algebra where $|\mathcal{V}_n| = n \in \mathbb{N}$ and $\mathcal{O}_n$ is a set of finitary operations on $\mathcal{V}_n$.

Definition 6.7. We will be reserving the notion of functional completeness for logics (i.e. algebras). When we are dealing with a theory **Th** that is functionally complete, we will say that **Th** is *expressively complete.*

Remark 6.8. As can be easily noted, if a logic is functionally complete it is truth complete. However, the converse does not necessarily hold. Thus, if having an expressively complete theory is something we require in the route to semantic closure, then having a truth complete theory might not be enough for this purpose.

Remark 6.9. It might be illuminating to cite a case of a function that is not definable in a given logic, but is definable in a theory based on that logic. In fact, in our case, **MSC** is not able to define the constant function for $\frac{1}{2}$, but it turns out that $\mathbf{MSC_w}$+ can do this. To establish this fact, we shall notice that $\mathbf{MSC_w}$+'s Liar sentence p^* is such that it can only receive the value $\frac{1}{2}$ in every valuation. Thus, in the context of the theory $\mathbf{MSC_w}$+ (but not, in the context of the logic **MSC**) it can be thought of p^* as the formula that "defines" the constant function for $\frac{1}{2}$. This also allows us to prove that **MSC**+ is *not functionally complete.*[26]

Theorem 6.10 ([15])**.** *Every three-valued logic capable of expressing the unary operations $\star_1$, $\star_2$ and the binary operation $\star_3$ is functionally complete.*

A	$\star_1 A$
1	1
$\frac{1}{2}$	0
0	$\frac{1}{2}$

A	$\star_2 A$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$A \star_3 B$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

[26] Moreover, as noticed by an anonymous referee, the operation $\star_1$ is also not definable in **MSC** and this fact also witnesses that the logic **MSC** is not *functionally complete.*

Theorem 6.11 (Truth-Completeness). **MSC** *and, thus,* $\mathbf{MSC_w}$+ *are truth complete.*[27]

Proof. The operators J_1, $J_{\frac{1}{2}}$ and J_0 are definable in **MSC+** as follows

- $J_1(A) =_{def} A \wedge \circ A$
- $J_{\frac{1}{2}}(A) =_{def} \neg\circ A$
- $J_0(A) =_{def} \neg A \wedge \circ A$

□

Theorem 6.12 (Expressive-Completeness). $\mathbf{MSC_w}$+ *is expressively complete*

Proof. Notice that $\mathbf{MSC_w}$+ can define the truth functional constants ⊤, ⊥, and has also a formula –e.g. The Liar sentence p^*– that receives the value $\frac{1}{2}$ in every valuation, which therefore can be used as a constant for the truth-value $\frac{1}{2}$. We can, thus, show now that $\mathbf{MSC_w}$+ can define the functions of [15] as follows

- $\star_1(A) =_{def} (p^* \to A) \wedge \circ A$
- $\star_2(A) =_{def} \neg A$
- $A \star_3 B =_{def} A \wedge B$

□

7. Conclusion

In the previous sections we showed that if getting functionally complete naïve theories of truth is as a *necessary* step in the route towards semantic closure, then it is possible to get non-trivial theories of this sort. If aiming at semantic closure is something that will result to be impossible in the future, it will not be because naïve theories of truth cannot accommodate the expressive power necessary to develop this enterprise. In fact, we showed (following closely a suggestion made by Laura Goodship in [13]) that all that is required to have naïve theories of truth that are as expressively rich as one wants them to be, is to express self-reference in a *weak* way (i.e. via equivalences) and not in a *strong* way (i.e. via identities).

It might be reasonable to ask: if we use a *weak* self-referential procedure to show that $\mathbf{MSC_w}$+ is nontrivial, could not we just construct an equally strong naïve theory of truth with e.g. $\mathbf{LP}^\circ$ and have a resulting theory $\mathbf{LP}^\circ_\mathbf{w}$+? This is actually the case, indeed, at least if we make suitable arrangements –e.g. using a similar weak self-referential procedure, resembling the one discussed in Section 5. In that case, it will not be difficult to run a non-triviality proof for e.g. $\mathbf{LP}^\circ_\mathbf{w}$+ along the same lines of the one presented above. Let us comment that, in any case, this seems not to be the preferred way of Graham Priest who in e.g. [22] chooses to drop Transparency instead of turning to express self-reference with a non-detachable conditional.

Nevertheless, our main concern here is not to pick naïve theory of truth to which we should commit ourselves, but to show that the route to semantic closure is not entirely blocked. Further applications of the approach presented here can be surely carried out, and we hope to analyze them in the near future.

The framework discussed here is, moreover, open to further expansions and developments. Two of the most interesting tasks ahead are to expand the language

[27] Thanks to an anonymous referee for discussion on this matter.

with more naïve semantic notions –for example, a *Validity predicate*– without trivializing the system. Whether or not applying the weakened self-referential procedure to these systems will allow us to have non-trivial systems is a question that we leave for further research.

Appendix: The Completeness Proof

We will use the method of reduction trees,[28] that allows to build for any given sequent, either a proof of that sequent, or a counterexample to it. The method also provides of a way of building the eventual counterexample. We will introduce the notions of subsequent and sequent union, that will be used in the proof:

Definition 7.1. A sequent $S = \Gamma \mid \Sigma \mid \Delta$ is a *subsequent* of a sequent $S' = \Gamma' \mid \Sigma' \mid \Delta'$ (written $S \sqsubseteq S'$) if and only if $\Gamma \subseteq \Gamma'$, $\Sigma \subseteq \Sigma'$, and $\Delta \subseteq \Delta'$.

Definition 7.2. A sequent $S = \Gamma \mid \Sigma \mid \Delta$ is the *sequent union* of a set of sequents $[\Gamma_i \mid \Sigma_i \mid \Delta_i]_{i \in I}$ (written $S = \sqcup[\Gamma_i \mid \Sigma_i \mid \Delta_i]_{i \in I}$) if and only if $\Gamma = \cup_{i \in I}\Gamma_i$, $\Sigma = \cup_{i \in I}\Sigma_i$ and $\Delta = \cup_{i \in I}\Delta_i$.

The construction starts from a root sequent $S_0 = \Gamma_0 \mid \Sigma_0 \mid \Delta_0$, and then builds a tree in stages, applying at each stage all the operational rules that can be applied, plus Derived Cut "in reverse", i.e. from the conclusion sequent to the premise(s) sequent(s).[29] Thus, stage 0 will just be the root sequent S_0. If it is an axiom, the branch is closed. For any stage $n + 1$, one of two following things might happen:

(1) For all branches in the tree after stage n, if the tip is an axiom, the branch is closed.
(2) For open branches: For each formula A in a sequent position in each open branch, if A already occurred in that sequent position in that branch (i.e. A has not been generated during stage $n + 1$), and A has not already been reduced during stage $n + 1$, then reduce A as is shown below. There are three possible positions in which a formula can appear in a sequent: either on (i) the left side, or on (ii) the middle, or on (iii) the right side. We need to consider all these possible cases.
 - If A is a negation $\neg B$, then: if A is in the left/ middle/ right position, extend the branch by copying its current tip and adding B to the right/ middle/ left position.

[28]For similar proofs, see [1], were these kind of reduction trees proofs for many-valued theories where originally presented, but also see [26] and [21].

[29]For the proof, we use an enumeration of the formulae and an enumeration of names. We will reduce, at each stage, all the formulae in the sequent, starting from the one with the lowest number, then continuing with the formula with the second lowest number, and moving on in this way until the formula with the highest number in the sequent is reduced. In case a formula that appears in more than one side of the sequent, we will start by reducing the formula that appears on the left side and then proceed to the middle and the right side, respectively. The final step, at each stage n of the reduction process, will be an application of the Derived Cut rule to the n-formula in the enumeration. If we apply a multi-premise rule, we will generate more branches that will need to be reduced. If we apply a single-premise rule, we just extend the branch with one more leave. We will only add formulae at each stage, without erasing any of them. As a result of the process just described, every branch will be ordered by the subsequent relation. Any branch that has an axiom as it topmost sequent will be closed. A branch that is not closed is considered open. This procedure is repeated until every branch is closed, or until there is an infinite open branch. If every branch is closed, then the resulting tree itself is a proof of the root sequent. If there is an infinite open branch Y, we can use it to build a counterexample to the root sequent.

- If A is a conjunction $B \wedge C$, then: (i) if A is in the left position, extend the branch by copying its current tip and adding both B and C to the left position. (ii) If A is in the middle position, split the branch in three: extend the first by copying the current tip and adding B to both the middle and right positions; extend the second by copying the current tip and adding C to the middle and right positions; and extend the third by copying the current tip and adding both B and C to the middle position. (iii) If A is in the right position, split the branch in two: extend the first by copying the current tip and adding B to the right position; and extend the second by copying the current tip and adding C to the right position.
- If A is a consistency assertion $\circ B$, then: (i) if A is in the left position, extend the branch by copying its current tip and adding B to the middle position. (ii) If A is in the right position, extend the branch by copying its current tip and adding B to the right and left positions. (iii) If A is in the middle position, then do nothing.
- If A is a classical negation assertion $\sim B$, then: (i) if A is in the left position, extend the branch by copying its current tip and adding B to the middle and the right positions. (ii) If A is in the right position, extend the branch by copying its current tip and adding B to the left position. (iii) If A is in the middle position, then do nothing.
- If A is a conditional $B \to C$, then: (i) if A is in the left position, split the branch in two: extend the first by copying the current tip and adding B to right position, and extend the second by copying the current tip and adding C to the left position. (ii) If A is in the middle position, split the branch in two: extend the first by copying the current tip and adding B to middle position, and extend the second by copying the current tip and adding C to the left position. (iii) If A is in the right position, extend the branch by copying the current tip, adding B to the left position, and adding C to the middle and right positions.

We will also apply the Derived Cut rule at each step. Consider the nth formula in the enumeration of formulae and call it A. Now extend each branch using the Derived Cut rule. For each open branch, if its tip is $\Gamma \mid \Sigma \mid \Delta$, split it in three and extend the new branches with the sequents $\Gamma, A \mid \Sigma, A \mid \Delta$, $\Gamma, A \mid \Sigma \mid \Delta, A$, and $\Gamma \mid \Sigma, A \mid \Delta, A$, respectively.

Now we need to repeat this procedure until every branch is closed, or, if that does not happen, until there is an infinite open branch. If the first scenario is the actual one, then the tree itself is a proof of the root sequent, because each step will be the result of an application of a structural or operational rule to the previous steps. If the second scenario is the actual one, we can use the infinite open branch to build a counterexample.

If in fact there is an infinite open branch Y, then the Derived Cut rule will have been used infinitely many times. Thus, every formula will appear at some point in the branch for the first time, and will remain in every step afterwards. Now, we first collect all sequents of the infinite open branch Y into one single sequent $S_\omega = \Gamma_\omega \mid \Sigma_\omega \mid \Delta_\omega = \sqcup\{S \mid S \text{ is a sequent of } Y\}$. Notice that, as Derived Cut has been applied infinitely many times in the construction of the branch, every formula will occur in exactly two places in S_ω.[30] Thus, there will be a valuation such that no formula in the sequent gets the value associated with the place where it occurs (i.e. 0 if the formula occurs in the left, $\frac{1}{2}$ if it occurs in the middle, 1 if it occurs in the right). Hence, for each formula A in the sequent, v will give to A a value different from the ones corresponding to the sides where A appears in the sequent. But that includes all the formulae in the initial and finite sequent S_0. That valuation, then, will also be a counterexample to S_0. Therefore that valuation will be a counterexample to the sequent being considered.

Thus, for atomic formulae A (propositional letters and truth assertions), $v(A) = 0$ or $\frac{1}{2}$ or 1, respectively, if and only if A does not appear in Γ_ω or Σ_ω or Δ_ω, respectively. Let us now take A to be any formula whatsoever.[31]

The rules for reducing formulae can be used to show by induction that, if none of the components of complex formulae (non-classical and classical negations, conjunctions, conditionals, consistency assertions) receive the value associated with any place in which they appear in S_ω, neither will the compound. We will not see, due to limitations of space, how this method works in detail. For conjunctions and paraconsistent negations, we proceed exactly as is shown in [26]. New cases are the cases of consistency assertions, classical negations and our peculiar conditionals. As the first two kinds of reduction behave very similarly to the cases of the other monadic constants, we will just check one type of reduction of a conditional assertion.

Let us now focus on to the cases of conditionals of the form $B \to C$. We need to consider three possible situations: (i) either the conditional appears in both the left and the right sides, or (ii) it appears both in the left and in the middle sides, or (iii) it appears on the middle and the right sides. We will just check what happens with case (i), and leave (ii) and (iii) to the reader.

In this case, eventually, $B \to C$ will be reduced from a sequent like $\Gamma, B \to C \mid \Sigma \mid \Delta, B \to C$. The reduction of the conditional on the right side will demand to copy the current tip, and also the addition of B in the left, and the addition of C in both the middle and the right sides of the sequent. But, as $B \to C$ appears also in the left side, this demands to split the branch in two, and to extend the first by copying the current tip and adding B to the right position, and also to extend the second by copying the current tip and adding C to the left position. As we have established, we need to reduce first the occurrences of the formula that appears on the left, then the ones that appears on the middle, and finally the ones that appears of the right side of the sequent. (Recall that we are talking about occurrences of the same formula, and which is the order to reduce them at some particular stage n.) The result of reducing first the sequent of the left, and then the one of the right, will be the result of splitting the branch in two, and (1) extending the first

[30] It cannot occur in the three places, because then there will be some finite stage n where the formula appears for the first time in the branch in the three sides. But then that sequent will be an axiom, and therefore the branch will be closed.

[31] Does A appear in exactly the places where $Tr(\langle A \rangle)$ appears? Yes. As any formula in sequent that corresponds to an infinite open branch, A appears in exactly two places. If $Tr(\langle A \rangle))$ appears in the only place where A does not appear, then, as $Tr(\langle A \rangle)$ will eventually be reduced, A will appear in the only place where it does not appear until that moment in branch. But then that sequent will be an axiom, and thus the branch will be closed. This is the only possibility that we need to consider. $Tr(\langle A \rangle)$ can not appear in less places that A: as any formula in a sequent corresponding to an infinite open branch, it has to appear in exactly two places.

by copying the current tip and adding B to the left and right position, but also add C to the middle and right position, and (2) extending the second by copying the current tip and adding C to the left, middle and right position, and also B to the left and right positions. Thus, the two new sequents will be:

$$\dfrac{\Gamma, B \to C, B \mid \Sigma, C \mid \Delta, B \to C, B, C}{\vdots} \qquad \dfrac{\Gamma, B \to C, B, C \mid \Sigma, C \mid \Delta, B \to C, C}{\vdots}$$

Thus, these are two new branches. The second one, will be an axiom, because the formula C appears in the three sides of the sequent, and then that particular branch will be closed. But that does not happen with the other branch. The complexity of B and C is less than the complexity of $B \to C$, hence the inductive hypothesis can be applied to them. Therefore, B will get value $\frac{1}{2}$, C will get value 0, and then $B \to C$ will get value $\frac{1}{2}$. Thus, none of these formulae in the branch receives the value associated with the sides of the sequent in which they appear.

By completing the induction along these lines, we can show that we can construct a valuation such that no formula receives the value associated with any place where it appears in S_ω. But, as we know, that includes all the formulae in the initial and finite sequent S_0. That valuation, then, will also be a counterexample to S_0, which is what we were looking for. Thus, for any sequent S, either it has a proof or it has a counterexample.

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